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SMALL SALEM NUMBERS, EXCEPTIONAL UNITS, AND LEHMER'S CONJECTURE

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ABSTRACT. Lehmer's conjecture says that there is an $\varepsilon > 0$ so that if an algebraic integer α is not a root of unity, then its Mahler measure $M(\alpha)$ is greater than $1+\varepsilon$. This suggests that if $M(\alpha) > 1$ is small, then α should behave like a root of unity. For example, there might be many small values of n such that $1-\alpha^n$ is a unit; that is, such that α^n is an exceptional unit.

The smallest Mahler measures currently known occur for Salem numbers, and Boyd has constructed a table of small Salem numbers. We verify experimentally that many powers of the numbers in Boyd's table are exceptional units. We also show that if α is an algebraic integer of degree d, then at most $O(d^{1+\varepsilon})$ powers of α can be exceptional units. Finally, we consider the Mahler measure (canonical height) associated to arbitrary rational maps $\phi(x)$ and raise some questions related to ϕ-Salem numbers and the ϕ-Lehmer conjecture.

1. Heights and Mahler measure. Recall that the *Mahler measure* of an algebraic integer α is the quantity $M(\alpha)$ defined by

$$M(\alpha) = \prod_{\sigma:\mathbf{Q}(\alpha)\hookrightarrow\mathbf{C}} \max\{|\sigma\alpha|, 1\}.$$

Here the product is over all of the embeddings of $\mathbf{Q}(\alpha)$ into $\mathbf{C}$. Clearly we always have $M(\alpha) \geq 1$, and an elementary result of Kronecker tells us when there is equality.

Theorem (Kronecker [**8**]). *$M(\alpha) = 1$ if and only if α is a root of unity.*

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