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TRIANGULAR NUMBERS AND ELLIPTIC CURVES

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Dedicated to Wolfgang Schmidt for his 60th birthday

ABSTRACT. Some arithmetic of elliptic curves and theory of elliptic surfaces is used to find all rational solutions (r, s, t) in the function field $\mathbf{Q}(m, n)$ of the pair of equations

$$r(r+1)/2 = ms(s+1)/2$$
$$r(r+1)/2 = nt(t+1)/2.$$

It turns out that infinitely many solutions exist. Several examples will be given.

1. Introduction. A triangular number Δ_r is by definition the sum

$$1 + 2 + \cdots + r = r(r+1)/2$$

of the first r natural numbers. Many properties of the triangular numbers

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45,\ 55, \ldots$$

have been discovered by Legendre, Gauss, Euler and others. For example, Legendre proved that a triangular number $\Delta_r > 1$ cannot be a cube, nor a fourth power. Gauss showed that every natural number is a sum of at most three triangular numbers. Euler proved that there are infinitely many squares among the triangular numbers, and he determined all of them. Euler also showed that infinitely many pairs (Δ_r, Δ_s) exist for which $\Delta_r = 3\Delta_s$. In fact, using the well-known theory of the Pell-Fermat equation $X^2 - dY^2 = 1$, it is not hard to show that for any natural number m there are infinitely many pairs (Δ_r, Δ_s) satisfying $\Delta_r = m\Delta_s$ (cf. [**2**]). On the other hand, using a result of Mordell on integral points on curves of genus 1 [**4**] it is shown

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