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DISTANCE FUNCTIONS ON VARIETIES OVER NON-ARCHIMEDIAN LOCAL FIELDS

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Let K be a field, complete with respect to an absolute value $|.|$. Let X be an algebraic variety defined over K. Then $X(K)$ has a natural topology coming from the topology of K induced by $|.|$ and it is possible, in many different ways, to describe this topology by a metric. This has been studied, for instance, in [**11**], where many functorial properties are obtained. However, the main focus of [**11**] is to obtain global height functions, so the metrics considered are defined in completions of global fields, and care was used to study how things varied with respect to the place. Also, archimedian valuations are considered. All this means that the results of [**11**] are stated as holding only modulo bounded functions. Some of the results of [**11**] were extended in [**1**]. Another approach, for projective space, was suggested in [**7**] and studied further in [**3**]. Again, the focus was on global fields. The purpose of this note is to concentrate on the non-archimedian local case and obtain more refined results. We will use a different development of the theory, but we will show that we recover the metrics defined by the aforementioned authors, in particular showing that they coincide, which is not obvious from their definitions. Our results will be sharp and will not involve any unspecified bounded function. Most, if not all, of our results will not be a surprise to the experts and, in fact, we have implicitly used these results, e.g. in [**13**]. Nevertheless, it seems appropriate to record these results with proofs, since no other source is currently available in the literature. At the end we will discuss some global results and give a sharpening of a result of Carlitz which suggests an interesting conjecture.

In the case $K = \mathbf{Q}_p$, the definition of our metric will be, informally, that $d(P, Q) = p^{-m}$, if P and Q are equal modulo p^m but not modulo p^{m+1}. This is more conveniently dealt with in the language of schemes. Perhaps with a bit more effort one could dispense with that.

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