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# A NEW REFINEMENT OF THE ARITHMETIC MEAN–GEOMETRIC MEAN INEQUALITY

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In what follows, we denote by $A_n$ and $G_n$ the weighted arithmetic and geometric means of the positive real numbers $x_1, \dots, x_n$, that is,

$$A_n = \sum_{i=1}^{n} p_i x_i \quad \text{and} \quad G_n = \prod_{i=1}^{n} x_i^{p_i},$$

where $p_1, \dots, p_n$ are nonnegative real numbers with $\sum_{i=1}^{n} p_i = 1$.

The famous arithmetic mean–geometric mean inequality $G_n \leq A_n$ has found much attention among many mathematicians, and numerous proofs, refinements, extensions and related results of what E.F. Beckenbach and R. Bellman call "probably the most important inequality, and certainly a keystone of the theory of inequalities," [**1**, p. 3], can be found in the literature. We refer to the monographs [**1, 2, 5, 6**] and the references therein.

In 1978, D.I. Cartwright and M.J. Field [**3**] proved the following interesting sharpening of the arithmetic mean–geometric mean inequality.

$$\frac{1}{2} \frac{1}{\max_{1 \leq i \leq n} x_i} \sum_{i=1}^{n} p_i (x_i - A_n)^2 \leq A_n - G_n, \tag{1}$$

with equality holding if and only if the $x_i$'s corresponding to positive $p_i$'s are all equal. Moreover, the authors pointed out that the constant $1/(2 \max_{1 \leq i \leq n} x_i)$ is best possible.

The aim of this paper is to show that inequality (1) remains valid if we replace on the lefthand side of (1) $A_n$ by $G_n$. Since

$$0 \leq (A_n - G_n)^2 = \sum_{i=1}^{n} p_i (x_i - G_n)^2 - \sum_{i=1}^{n} p_i (x_i - A_n)^2,$$

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