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CONSTRUCTION OF INDECOMPOSABLE HERONIAN TRIANGLES

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ABSTRACT. We give a simple characterization in terms of the tangents of the half-angles of a primitive Heronian triangle for the triangle to be decomposable into two Pythagorean triangles. This characterization leads to easy constructions of Heronian triangles not so decomposable.

1. Introduction. The ancient formula attributed to Heron of Alexandria on the area of a triangle in terms of the lengths of the sides a, b, c, and the semi-perimeter $s = (a+b+c)/2$,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \tag{1}$$

naturally suggests the problem of constructing triangles with integer sides and integer areas. We shall call such triangles *Heronian*, and denote one such triangle with sides a, b, c and area Δ by $(a, b, c; \Delta)$. A common construction of Heronian triangles is to juxtapose two Pythagorean triangles (right triangles with integer sides) along a common leg. See, for example, Dickson [**5**, Chapter 5] and Thébault [**8**]. Indeed, a triangle is Heronian if and only if it is the juxtaposition of two Pythagorean triangles along a common leg, or a reduction of such a juxtaposition (Carlson [**2**], with correction in Singmaster [**7**]). Cheney [**4**] has given a construction of Heronian triangles in terms of positive rational numbers t_1, t_2, t_3, satisfying

$$t_1t_2 + t_2t_3 + t_3t_1 = 1. \tag{2}$$

Here t_i, $i = 1, 2, 3$, are the tangents of the half-angles of the triangles, and any two of them determine the third. Lehmer [**6**] had noted that a triangle with rational sides has rational area if and only if the tangents of its half-angles are all rational. Such triangles are called *rational*. The similarity class of a rational triangle contains triangles with integer sides. Such triangles are necessarily Heronian, see Proposition 1 below.

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