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UNSOLVABLE CASES OF $P^3 + Q^3 + cR^3 = dPQR$

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Introduction. Consider the diophantine equation

$$P^3 + Q^3 + cR^3 = dPQR \tag{1}$$

where c, d, P, Q and R are rational integers, all $\neq 0$. The first results were stated (without proofs) by Sylvester [**11**] and later the equation was treated by several authors [**9**], some of them using the fact that a nontrivial solution of (1) implies that $y^2 = x^3 + (dx + 4c)^2$ has a (rational) solution with $x \neq 0$. This connection with elliptic curves makes it possible to use the powerful computational techniques for deciding solvability, that are based on calculation of the appropriate L-series for a given c and d. Bremner and Guy's paper [**2**] on equation (1) with $c = 1$ demonstrates well the power of these techniques for determining the rank and finding solutions in particular cases. However, classical methods still have some advantages, e.g., they often cover an infinite number of parameter values and the actual testing is usually far easier to perform. They may also give complementary insights, e.g., in this paper a decisive cubic residue condition occurs in every case. Using classical methods, noticeable progress has been made regarding the case $c = f^k$ when f is a prime number and $3 \nmid k$ [**5, 9**], but very few generic results exist when c has several different prime factors. This illustrates the difficulties encountered when using classical methods, but in this paper we show that this path is not yet fully explored and give results that, together with earlier results, for $c = 1, f$, cover nearly all unsolvable cases for $c = 1, f, fg, f^2g, f^3, f^3g$ where f and g are different prime numbers.

For convenience, we make the following

Definitions. If f is a cubic residue of h, we use the notation $f \sim cr\, h$. Similarly, $f \sim cr\, h$ is used when f is a cubic nonresidue of h. The subclass $\mathbf{S}$ of (positive rational) primes is defined by $p \in \mathbf{S}$ if and

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