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AN INTRODUCTION TO ZARISKI SPACES OVER ZARISKI TOPOLOGIES

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ABSTRACT. Given a topology Ω on a set X, we consider a structure (Y, Γ) such that the relationship between (Y, Γ) and (X, Ω) is similar to the relationship between a module and its ring of scalars. Indeed, this structure is a module analogue of the Zariski topology on the prime spectrum of a ring R in that its construction uses the prime submodules of an R-module M in essentially the same way that the construction of the Zariski topology uses the prime ideals of R. It is shown that an R-module homomorphism f between two R-modules induces in a natural way a homomorphism between their associated structures, and in case f is an epimorphism, the induced homomorphism is continuous in nontrivial cases.

1. Zariski spaces. Throughout this paper R denotes a commutative ring with identity and M a unital R-module. If I is an ideal of R, we write $I \triangleleft R$, and $A \leq M$ means that A is a submodule of M. If $A \leq M$, then $(A : M)$ represents the ideal $\{r \in R : rM \subseteq A\}$.

A submodule P of M is called *prime* if P is proper, and whenever $rm \in P$, $r \in R$ and $m \in M$, then $m \in P$ or $r \in (P : M)$. The collection of all prime submodules of M is denoted by $\operatorname{spec} M$. If A is a submodule of M, then the *radical* of A, denoted $\operatorname{rad} A$, is the intersection of all prime submodules of M which contain A, unless no such primes exist, in which case $\operatorname{rad} A = M$. In fact, there exist modules M with no prime submodules at all, though any such module M could not be finitely generated. Such modules are called *primeless*. Studies of prime submodules can be found in [**1, 3, 5**] and [**7–12**], among others. In particular, one can find the following, easily proven but useful, result in [**5**] or [**7**].

Lemma 1. *Let P be a (proper) submodule of M. Then P is prime in M if and only if $(P : M)$ is prime in R and M/P is a torsion-free $R/(P : M)$-module.*

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