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A NOTE ON PI INCIDENCE ALGEBRAS

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ABSTRACT. The question when the incidence algebra of a locally finite partially ordered set over a commutative ring with identity is a polynomial identity algebra is discussed. When the incidence algebra is a PI algebra, all its minimal degree, linear, homogeneous identities are determined.

Let R be a commutative ring with identity and $R[X_1, X_2, \dots]$ the ring of polynomials in the noncommuting indeterminates $X_1, X_2, \dots$ over R. A polynomial of the form $rX_{i_1}^{e_1} X_{i_2}^{e_2} \cdots X_{i_n}^{e_n}$, with $r \in R$, i_j a positive integer and e_j a nonnegative integer for $j = 1, 2, \dots, n$, and $i_j \neq i_{j+1}$ for $j = 1, 2, \dots, n-1$, is called a monomial. It has coefficient r and degree $\sum_{j=1}^{n} e_j$. An element $f \in R[X_1, X_2, \dots]$ is called a polynomial. It is the sum of monomials and has degree, $\deg(f)$, the maximal degree of the monomials in f. If all monomials in f have the same degree, then f is a homogeneous polynomial. If none of the indeterminates X_i for $i > n$ appear in any monomial of f, we write $f = f(X_1, X_2, \dots, X_n)$. If A is an algebra over R, we write $p(r_1, r_2, \dots, r_n)$ for the element of R obtained when r_j is substituted for X_j, for $j = 1, 2, \dots, n$.

If A is an algebra over R, a nonzero polynomial $f = f(X_1, X_2, \dots, X_n) \in R[X_1, X_2, \dots]$ is an identity for A if for every $\alpha_1, \alpha_2, \dots, \alpha_n \in A$ we have $f(\alpha_1, \alpha_2, \dots, \alpha_n) = 0$. If, in addition, a monomial of f of highest degree has coefficient the identity of R, then f is a polynomial identity for A. The algebra A is a polynomial identity algebra, PI algebra, if A has a polynomial identity. If f is a polynomial identity for A, then from a result of Kaplansky [**4**], A satisfies a homogeneous polynomial identity of the form $X_1 X_2 \cdots X_n + p$, where p is the sum of monomials of the form $r_\sigma X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(n)}$, with $r_\sigma \in R$, σ a nonidentity element of the symmetric group S_n, and $n \leq \deg(f)$. The polynomial $s(X_1, X_2, \dots, X_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(n)}$ is called the standard polynomial of degree n. Here $\operatorname{sgn}(\sigma)$ denotes the sign of σ. The famous Amitsur-Levitzki theorem [**1**] tells us that

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