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ON BOUNDED VECTOR FIELDS

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ABSTRACT. We introduce the notion of a strongly bounded vector field, which is closely related to the usual notion of a bounded vector field, and we prove that any $\mathcal{C}^1$ strongly bounded vector field in $\mathbf{r}^n$ with finitely many critical points satisfies that the sum of the indices of the vector field at these critical points is equal to $(-1)^n$. In the planar case we improve this result since we prove it for $\mathcal{C}^1$ bounded vector fields. Moreover, when $n \geq 3$, we present examples of $\mathcal{C}^\infty$ bounded vector fields in $\mathbf{r}^n$, being obviously not strongly bounded, not satisfying that the sum of the indices at the critical points is $(-1)^n$.

1. Introduction. Let $X : U \to \mathbf{R}^n$ be a $\mathcal{C}^1$ vector field where U is an open set of $\mathbf{R}^n$, and let $\dot{x} = X(x)$ be the differential system associated to X. Consider $\varphi(x,t)$, the integral curve of X such that $\varphi(x,0) = x$, and let I_x be its maximal interval of definition. We say that X is a *bounded vector field* if for each $x \in U$ there exists some compact set $K \subset U$ such that $\varphi(x,t) \in K$ for all $t \in I_x \cap (0,+\infty)$.

It is a well-known fact, see [**10**], that if X is a bounded vector field, then the integral curve of any point is defined for all positive time and that the ω-limit of any point x, $\omega(x)$, is not empty and compact.

Bounded vector fields are interesting from the theoretical and practical point of view. Thus we can mention, for instance, previous approaches in [**2, 3**] and [**7**]. Given a vector field, it may be very difficult to know if it is bounded or not. In this setting it is interesting to give necessary and sufficient conditions in order to assure that a vector field is bounded. The goal of this paper is to generalize, as far as possible, a property which is satisfied by certain families of bounded vector fields. This property takes into account the index of the vector field at its critical points.

In what follows we will say that X is a *strongly bounded vector field*

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