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MATRICIAL RANGES OF QUADRATIC OPERATORS

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ABSTRACT. We show that if T is a quadratic operator on a Hilbert space, then (1) the numerical range of T is an (open or closed) elliptical disc (or its degenerate form) and (2) for every $n \geq 1$, the nth matricial range of T consists of $n \times n$ matrices whose numerical ranges are contained in the closure of the numerical range of T.

For a bounded linear operator T on a complex Hilbert space H its *numerical range* $W(T)$ is by definition the set $\{\langle Tx, x\rangle : x \in H \text{ and } \|x\| = 1\}$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in H. As is well known, to determine the numerical range of a general operator is a very difficult task. Toeplitz, in the earliest paper on this subject [**10**], did this for operators on a two-dimensional space: their numerical ranges are (closed) elliptical discs. One purpose of this paper is to show that an analogous result holds for quadratic operators on a Hilbert space. Recall that T is *quadratic* if it satisfies $T^2 + \lambda_1 T + \lambda_2 I = 0$ for some scalars λ_1 and λ_2. In contrast to the finite-dimensional case, numerical ranges of operators on an infinite-dimensional space are in general not closed. We will determine for quadratic operators when their numerical ranges are.

In the literature, there are miscellaneous generalizations of the numerical range. The one to the matricial range seems to be most natural and useful. Specifically, for every $n \geq 1$, the nth *matricial range* $W^n(T)$ of an operator T on H consists of $n \times n$ matrices of the form $\phi(T)$, where ϕ is a unital completely positive linear map from $\mathcal{B}(H)$, the C^*-algebra of all operators on H, to M_n, the C^*-algebra of $n \times n$ matrices. This was first introduced by Arveson [**2**]. As was shown by him, they provide complete unitary invariants for certain compact operators. Note

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