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A SUBGROUP OF THE GROUP OF UNITS IN THE RING OF ARITHMETIC FUNCTIONS

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In memory of Toni Dehn

0. Introduction. Å is the ring of arithmetic functions with convolution as multiplication. It is well known that Å is a unique factorization domain [**3**]. Its ideal structure has been studied by Shapiro [**5**]. The group of units, particularly the subgroup of multiplicative functions, has been investigated by many people over the years. The multiplicative functions can be characterized as those arithmetic functions which are completely determined by their values at prime powers. Among them are the *completely* multiplicative functions, namely, those that are characterized by their values at the primes. The subgroup of the group of multiplicative functions generated by the completely multiplicative functions, the (group of) *rational functions*, was studied in a paper of Carroll and Gioia [**2**]. The name *rational functions* is due to Vaidyanathaswamy [**6**, pp. 611–612]. It is this subgroup, denoted here by $M^{\blacksquare}$, that we are concerned with.

Among other results, we show that $M^{\blacksquare}$ is a free (abelian) group; in particular, it is torsion-free and each element has a unique representation in terms of a generating set consisting of completely multiplicative functions. The group $M^{\blacksquare}$ is especially rich in subgroups. Our general approach is to look for "interesting" subgroups, that is, we shall use the subgroup structure as a useful means of classifying the arithmetic functions in this group.

Let

$$M_k = \{\gamma \in M^{\blacksquare}; \gamma = \alpha * \cdots * \alpha, k \text{ times}, \alpha \in M_1\}$$

and

$$M_k^{\sim} = \{\gamma^{-1} \in M^{\blacksquare}; \gamma \in M_k\},$$

where M_1 is the set of completely multiplicative functions. Then every element of $M^{\blacksquare}$ can be written as a (convolution) product $\gamma_* * \gamma_j^{-1}$,

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