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THE DENSITY OF PRIMES P, SUCH THAT -1 IS A RESIDUE MODULO P OF TWO CONSECUTIVE FIBONACCI NUMBERS, IS $2/3$

CHRISTIAN BALLOT

ABSTRACT. Given $\eta_1, \eta_2 \in \{\pm 1\}$, we calculate the exact proportion of primes p such that η_1, η_2 appear consecutively as residues of the Fibonacci sequence modulo p.

Introduction. Let $\eta_1, \eta_2 \in \{\pm 1\}$. In this paper we compute the density of the set of primes p such that η_1, η_2 appear as consecutive residues of the Fibonacci sequence (F_n) modulo p, i.e., such that there exists $n \in \mathbf{N} : (F_n, F_{n+1}) \equiv (\eta_1, \eta_2) \pmod p$.

The method used originated with Hasse, but its scope was later extended by Lagarias, and then Ballot. Let $U = (U_n)_{n \geq 0}$ be a linear recurrence sequence with integral terms and characteristic polynomial $f(X) \in \mathbf{Z}[X]$. Hasse [**6**] showed that for binary recurrence sequences $U_n = a^n + 1$, $a \in \mathbf{Z}$, one could compute the precise density of primes p such that p divides U, i.e., such that there exists $n \in \mathbf{N}$, $p \mid U_n$. Lagarias [**7**] went further by proving that Hasse's method applied to some binary linear recurrence sequences whose characteristic polynomials had *irrational* roots, in particular, to $U_n = L_n$, the sequence of Lucas numbers. The present author [**1**] discovered that one could generalize the method to the computing of densities of prime divisors of some linear recurrence sequences of arbitrary order $m \geq 2$ as long as one defined division of U to mean p divides $m - 1$ consecutive terms of the sequence U. (We then say that p is a *maximal divisor of* U.) However, all sequences of order $m \geq 3$ to which the method was applied in the author's memoir [**1**] had characteristic polynomials with rational roots. Here, for the first time, we deal with a *ternary* recurrence sequence whose characteristic polynomial, namely $f(X) = (X - 1)(X^2 - X - 1)$, has some *irrational* roots. Thus, we

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