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CONFORMAL IMAGES OF TANGENTIAL AND NONTANGENTIAL ARCS

JOHN AKEROYD

If f is bounded and analytic in $\mathbf{D} := \{z : |z| < 1\}$ and $\lim_{r\to 1^-} f(re^{i\theta})$ exists for some θ, then, by a normal families argument, $f(z)$ approaches that radial limit as z in $\mathbf{D}$ approaches $e^{i\theta}$ along any nontangential path; see [**1**, Theorem 1.3, p. 6]. In this note we give an analogous result for functions that are analytic and univalent in $\mathbf{D}$; with no loss of generality, we let $e^{i\theta} = 1$ throughout. We first observe that, for any function f that is both analytic and univalent in $\mathbf{D}$, $f([0,1))$ is rectifiable if and only if $f(\gamma\backslash\{1\})$ is rectifiable for each rectifiable Jordan arc γ contained in $\mathbf{D}\cup\{1\}$ that has a nontangential approach in $\mathbf{D}$ to 1 and that satisfies a certain restriction on its "oscillations" near 1 (Theorem 1). We also show that if γ has a tangential approach in $\mathbf{D}$ to 1, then there is a Jordan region Ω and a conformal mapping φ from $\mathbf{D}$ to Ω such that $\varphi([0,1]) = [0,1]$ and yet $\varphi(\gamma)$ is not rectifiable (Theorem 2); for a related result, see [**5**].

To establish the terms of our discussion, let γ be a Jordan arc from $[0,1]$ to the complex plane $\mathbf{C}$ such that $\gamma([0,1))$ is contained in $\mathbf{D}$ and $\gamma(1) = 1$. If the limit as t approaches 1 of $(1-|\gamma(t)|)/(|1-\gamma(t)|)$ exists and is zero, then we say that γ has a *tangential approach* in $\mathbf{D}$ to 1. And, if there exists $\varepsilon > 0$ such that $\varepsilon \le (1-|\gamma(t)|)/(|1-\gamma(t)|)$ whenever $0 \le t < 1$, then we say that γ has a *nontangential approach* in $\mathbf{D}$ to 1. Throughout this paper we let γ denote both the Jordan arc and its *trace* $\gamma([0,1])$. Let $T(z) = (1-z)/(1+z)$ be the Möbius transformation that maps $\{z : \mathrm{Re}\,(z) > 0\}$ onto $\mathbf{D}$, 0 to 1 and 1 to 0. For each nonnegative integer n, let $a_n = T(2^{-n})$ $(= (2^n-1)/(2^n+1))$; notice that $\rho(a_n, a_{n+1}) = (1/3)$ for all n, where $\rho(z,w) := |(z-w)/(1-\bar{w}z)|$ is the *pseudohyperbolic* distance between the points z and w in $\mathbf{D}$. If γ is a rectifiable Jordan arc in $\mathbf{D}\cup\{1\}$, then, for $n = 0,1,2,\dots$, let $\gamma_n = \{z \in \gamma : a_n \le |z| < a_{n+1}\}$ and (with the reference to γ understood), let $M_n = \mathrm{length}\,(\gamma_n)/(a_{n+1}-a_n)$; $\mathrm{length}\,(\gamma_n) := \Lambda_1(\gamma_n)$—the one-dimensional Hausdorff measure of γ_n. For $0 < \varepsilon < 1$ and any

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