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MAXIMALITY OF THE HYPERCUBE GROUP

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ABSTRACT. In this paper we prove maximality of the hypercube group $\mathbf{B}_n \leqslant \mathbf{O}(n)$ for $n \geq 3$, $n \neq 4$, as a closed subgroup of $\mathbf{O}(n)$. $\mathbf{B}_4 \leqslant \mathbf{O}(4)$ is not maximal, but we are able to describe all closed supergroups of $\mathbf{B}_4$. Furthermore, we indicate how this result is used in bifurcation theory for $\mathbf{O}(n)$-equivariant equations like semilinear elliptic boundary value problems.

1. Introduction. In this paper we will discuss the symmetry group of the n-cube $[-1,1]^n \subset \mathbf{R}^n$, $n \geq 3$. We will denote this group by $\mathbf{B}_n$. The questions we are interested in are whether $\mathbf{B}_n \leqslant \mathbf{O}(n)$ is a maximal closed subgroup or, if not, which are the nontrivial closed supergroups of $\mathbf{B}_n$.

In Section 2 we prove maximality of the hypercube group $\mathbf{B}_n \leqslant \mathbf{O}(n)$ for $n \geq 3$, $n \neq 4$, in the sense that there is no nontrivial closed supergroup of $\mathbf{B}_n$ in $\mathbf{O}(n)$. $\mathbf{B}_4 \leqslant \mathbf{O}(4)$ is not maximal, but we are able to describe in Section 3 all closed supergroups of $\mathbf{B}_4$.

A first step in the proof is to show discreteness and hence finiteness of a supergroup Γ of $\mathbf{B}_n$. This follows basically from the fact that $\mathbf{B}_n$ acts irreducible on the Lie algebra of $\mathbf{O}(n)$. The finite group Γ is then set in relation to the reflection group guaranteed by reflections in Γ and their normalizer which, to the very end, determines Γ itself.

The method to determine the various normalizers is always very similar. Essentially all is based on the knowledge of a characteristic subgroup Z of the finite reflection group, say $G \leqslant \mathbf{O}(n)$. Denoting by $\mathcal{R}$ the set of roots of G we have Z acting on $\mathbf{R}\mathcal{R}$ (or a certain subset) in the natural way. Therefore, $\mathbf{R}\mathcal{R}$ decomposes in Z-orbits and elements of the normalizer of G now act on these orbits by permutation. This already enables computation of the normalizer, at least in our examples.

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