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PRODUCTS OF FACTORIALS IN BINARY RECURRENCE SEQUENCES

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ABSTRACT. In this paper, we show that every nondegenerate binary recurrence sequence contains only finitely many terms which can be written as products of factorials. Moreover, all such terms can be effectively computed. We also find all the terms of the Fibonacci sequence which are products of factorials.

1. Introduction. Let α and β be nonzero algebraic integers, and let a and b be nonzero algebraic numbers. For any integer $n \geq 0$, let

$$u_n = a\alpha^n + b\beta^n. \tag{1}$$

It is clear that

$$u_{n+2} = ru_{n+1} + su_n \quad \text{for } n = 0, 1, \dots, \tag{2}$$

where $r = \alpha + \beta$ and $s = -\alpha\beta$. We refer to the sequence $(u_n)_{n\geq 0}$ as a *binary recurrence sequence*. If u_0 and u_1 are algebraic integers, then $(u_n)_{n\geq 0}$ is a binary recurrence sequence of algebraic integers. The sequence $(u_n)_{n\geq 0}$ is said to be nondegenerate if α/β is not a root of unity. We refer to the equation

$$x^2 - rx - s = 0$$

as the *characteristic equation* of the sequence $(u_n)_{n\geq 0}$.

Let $\mathcal{PF}$ be the set of all positive integers which can be written as products of factorials; that is,

$$\mathcal{PF} = \left\{ w \mid w = \prod_{j=1}^{k} m_j! \quad \text{for some } m_j \geq 1 \right\}. \tag{3}$$

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