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RANK 2 VECTOR BUNDLES IN A NEIGHBORHOOD OF AN EXCEPTIONAL CURVE OF A SMOOTH SURFACE

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ABSTRACT. Let $D \cong \mathbf{P}^1$ be an exceptional divisor on the smooth surface W and U the formal neighborhood of D in W. Let E be a rank 2 vector bundle on U. Here we associate to E an integer $t \geq 1$, a finite family E_i, $1 \leq i \leq t$, of rank 2 vector bundles on U and a finite sequence $\{(a_i, b_i)\}_{1 \leq i \leq t}$ of pairs of integers such that $E_i|D$ has splitting type (a_i, b_i), $E_1 = E$, $a_t = b_t$, $a_{i+1} + b_{i+1} = a_1 + b_1 + i$ and $b_i < b_{i+1} \leq a_{i+1} \leq a_i$ for $2 \leq i \leq t$. Vice versa, for any such sequence we prove the existence of at least one such bundle. We compute the second Chern class of E in terms of $\{(a_i, b_i)\}_{1 \leq i \leq t}$ and show that $\mathbf{O}_U(-a_1 D) \oplus \mathbf{O}_U(-b_1 D)$ is the unique bundle with splitting type (a_1, b_1) and maximal c_2.

0. Introduction. Let W be either a smooth connected quasi-projective surface defined over an algebraically closed field or a smooth connected two-dimensional manifold. We assume that W contains an exceptional divisor D, i.e., a smooth curve $D \cong \mathbf{P}^1$ with $\mathbf{O}_D(-1)$ as a normal bundle. Let U be either the formal completion of W along D or, if we work over $\mathbf{C}$, a small tubular neighborhood of D in W for the Euclidean topology. Let $\mathbf{I}$ be the ideal sheaf of D in U. Let E be a rank two vector bundle on U and (a, b) be the splitting type of $U|D$, i.e., let a, b be the integers with $a \geq b$ and such that $E|D \cong \mathbf{O}_D(a) \oplus \mathbf{O}_D(b)$. In the introduction of this paper we will associate to E an integer $t \geq 1$ and a finite sequence $\{(a_i, b_i)\}_{1 \leq i \leq t}$ of pairs of integers with $a_1 = a$, $b_1 = b$, $a_t = b_t = (a + b + t - 1)/2$ and a finite number of bundles E_i, $1 \leq i \leq t$, with $E_1 = E$, $E_i|D$ with splitting type (a_i, b_i).

Remark 0.1. It is well known and easy to check that if $E|D$ is trivial, then E is trivial. Furthermore, if W is quasi-projective, $E|D$ is trivial and $E \cong F|U$ with F algebraic vector bundle on W, then there exists a Zariski open neighborhood V of D with $F|V \cong \mathbf{O}_V^{\oplus 2}$ (use for

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