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QUASI-PURIFIABLE SUBGROUPS AND HEIGHT-MATRICES

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ABSTRACT. Let G be an arbitrary abelian group. A subgroup A of G is said to be quasi-purifiable in G if a pure subgroup H of G exists containing A such that A is almost-dense in H and H/A is torsion. Such a subgroup H is called a quasi-pure hull of A in G. First we prove that a torsion-free rank-one subgroup A of G is quasi-purifiable in G if and only if, for every prime p and every $a \in A$, $h_p(a) \geqq \omega$ implies $h_p(a) = \infty$. Next we use the result to compute the height-matrix of the torsion-free element a of an abelian group whose torsion part $T(G)$ is torsion-complete, then all torsion-free rank-one subgroups of G are quasi-purifiable in G and hence the height-matrices of the torsion-free elements of the group G can be computed.

1. Introduction. Let p be a prime. A subgroup A of an arbitrary abelian group G is said to be *p-purifiable* (*purifiable*) in G if a p-pure (pure) subgroup H of G containing A which is minimal among the p-pure (pure) subgroups of G that contain A. Such a subgroup H is said to be a *p-pure hull* (*pure hull*) of A in G.

Hill and Megibben [**7**] established properties of pure hulls of p-groups and characterized the p-groups for which all subgroups are purifiable.

Later, Benabdallah and Irwin [**2**] introduced the concept of almost-dense subgroups of p-groups and used it to determine the structure of pure hulls of purifiable subgroups of p-groups.

Furthermore, Benabdallah and Okuyama [**3**] introduce new invariants, the so-called nth *overhangs* of a subgroup of a p-group, which are related to the nth relative Ulm-Kaplansky invariants. Using them, they obtained a necessary condition for subgroups of p-groups to be purifiable.

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