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COMMUTATIVE ENDOMORPHISM ALGEBRAS OF TORSION-FREE, RANK-TWO KRONECKER MODULES WITH SINGULAR HEIGHT FUNCTIONS

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Dedicated to Professor J.D. Reid on the occasion of his retirement

ABSTRACT. As is the case with Abelian groups, rank-1, torsion-free Kronecker modules are characterized by height functions. A height function is called singular provided it never assumes the value infinity. The endomorphism algebras of singular, rank-1, torsion-free Kronecker modules are trivial. Here we consider the endomorphism algebras of torsion-free, rank-2 modules that are extensions of finite-dimensional modules by modules of rank-1. If K is the ground field and $K(X)$ the field of rational functions, the endomorphism algebras of the rank-2, indecomposable modules are known to be commutative K-subalgebras of the matrix ring $M_2(K(X))$. When the height function is nonsingular the resulting endomorphism algebras can be varied, including, for example, coordinate rings of elliptic curves. This paper examines the possibilities for the singular case, which we show are more limited. Yet their endomorphism algebras offer examples from an important class of commutative rings, namely, zero-dimension, local rings.

1. Introduction. For an algebraically closed field K, a function $h : K \cup \{\infty\} \rightarrow \{\infty, 0, 1, \dots\}$ is called a *height function*. Height functions are just as pervasive in the theory of Kronecker modules as they are in the theory of abelian groups [**4, 7**]. Every $K[X]$-module may be considered a Kronecker module, see [**5**]. The concepts used in Abelian groups have seen fruitful extension to Kronecker modules. In turn, Kronecker modules have helped guide the development of the general representation theory of finite-dimensional algebras, posets and indirectly Abelian groups, see, e.g., [**1, 2, 8, 9, 14**].

The Kronecker modules that are torsion-free extensions of finite-dimensional, rank-1 modules by infinite-dimensional, rank-1 modules have no analogue in $K[X]$-module theory. These modules, which are

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