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LOCALLY INJECTIVE MODULES AND LOCALLY PROJECTIVE MODULES

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ABSTRACT. Our dual notions "locally injective" and "locally projective" modules in Mod-R are good tools to study the relations between the singular, respectively cosingular, submodule of $\mathrm{Hom}_R(M, W)$ and the total $\mathrm{Tot}\,(M, W)$. These notions have further interesting properties.

1. Introduction. For a ring R with $1 \in R$ we denote by Mod-R the category of all unitary R-right modules. If A is a submodule of the module M, then $A \subseteq^0 M$, respectively $A \subseteq^* M$, denotes that A is a small, or superfluous, respectively a large or essential, submodule of M. Further $A \subseteq^\oplus M$ means that A is a direct summand of M. We have to use the following fundamental lemma.

Lemma 1.1. *For* $f \in \mathrm{Hom}_R(W, M)$ *the following conditions are equivalent*

(i) *There exists* $g \in \mathrm{Hom}_R(W, M)$ *such that* $e := gf = e^2 \neq 0$ *(*e *is an idempotent in* $\mathrm{End}\,(M)$*).*

(ii) *There exists* $h \in \mathrm{Hom}_R(W, M)$ *such that* $d := fh = d^2 \neq 0$ *(*d *is an idempotent in* $\mathrm{End}\,(W)$*).*

(iii) *There exist direct summands* $0 \neq A \subseteq^\oplus M$*,* $B \subseteq^\oplus W$*, such that the mapping* $A \ni a \mapsto f(a) \in B$ *is an isomorphism.*

For the proof, and for the proof of the following lemma, see [**4**]. If the conditions of the lemma are satisfied for f, we say that f is *partially invertible* (abbreviated 'pi'). The *total* of M, W, denoted by

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When I gave the talk at the Honolulu Conference I did not know that the notation "locally projective" was already used by Birge Zimmermann in her paper, *Pure submodules of direct products of free modules*, Math. Ann. **224** (1976), 233–245. I regret this very much and hope that this remark will help to avoid confusion.