ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 4, Winter 2002

ON KERVAIRE AND MURTHY'S CONJECTURE

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ABSTRACT. Let p be a semi-regular prime, let C_{p^n} be a cyclic group of order p^n and let ζ_n be a primitive p^{n+1}th root of unity. There is a short exact sequence

$$0 \to V_n^+ \oplus V_n^- \to \operatorname{Pic} \mathbf{Z}C_{p^{n+1}} \to \operatorname{Cl} \mathbf{Q}(\zeta_n) + \operatorname{Pic} \mathbf{Z}C_{p^n} \to 0.$$

In 1977 Kervaire and Murthy established an exact structure for V_n^-, proved that $\operatorname{Char}(V_n^+) \subseteq \operatorname{Char}(\mathcal{V}_n^+) \subseteq \operatorname{Cl}^{(p)}(\mathbf{Q}(\zeta_{n-1}))$, where V_n is a canonical quotient of $\mathcal{V}_n$ and conjectured that $\operatorname{Char}(V_n^+) \cong (\mathbf{Z}/p^n\mathbf{Z})^r$ where r is the index of irregularity of p.

We prove that, under a certain extra condition on p, $\mathcal{V}_n \cong \operatorname{Cl}^{(p)}(\mathbf{Q}(\zeta_{n-1})) \cong (\mathbf{Z}/p^n\mathbf{Z})^r$ and $V_n \cong \bigoplus_{i=1}^r (\mathbf{Z}/p^{n-\delta_i}\mathbf{Z})$, where δ_i is 0 or 1.

1. Introduction. Let p be an odd semi-regular prime, let C_{p^n} be the cyclic group of order p^n and let ζ_n be a primitive p^{n+1}th root of unity. For $k \geq 0$ and $i \geq 1$, let $A_{k,i} := \mathbf{Z}[x]/((x^{p^{k+i}}-1)/(x^{p^k}-1))$ and $D_{k,i} := A_{k,i} \bmod p$. Note that $A_{n,1} \cong \mathbf{Z}[\zeta_n]$. By a generalization of Rim's theorem (see for example [**1**]), $\operatorname{Pic} \mathbf{Z}C_{p^n} \cong \operatorname{Pic} A_{0,n}$ for all $n \geq 1$. It is well known that there exists a pull-back diagram (Cartesian square)

$$\begin{array}{ccc} A_{0,n+1} & \longrightarrow & \mathbf{Z}[\zeta_n] \\ \downarrow & & \downarrow \\ A_{0,n} & \longrightarrow & D_{0,n} := \dfrac{A_{0,n}}{pA_{0,n}} \end{array}$$

and an associated Mayer-Vietoris exact sequence

$$\mathbf{Z}[\zeta_n]^* \oplus A_{0,n}^* \to D_{0,n}^* \to \operatorname{Pic} A_{0,n+1} \to \operatorname{Pic} \mathbf{Z}[\zeta_n] \oplus \operatorname{Pic} A_{0,n} \to \operatorname{Pic} D_{0,n}.$$

1991 AMS *Mathematics Subject Classification.* 11R65, 11R21, 19A31.
Key words and phrases. Picard groups, integral group rings.
Received by the editors on August 7, 2001, and in revised form on