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# PURE PROJECTIVES AND INJECTIVES

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ABSTRACT. A module over a ring $R$ is pure projective or pure injective if it has the projective or injective property relative to all pure short exact sequences. Here a more restrictive concept of purity is introduced which singles out a certain subset of the pure short exact sequences. The modules which have the projective and injective property relative only to this smaller subset are studied.

**1. Introduction.** For infinite cardinals $\mu, \aleph$, $(\mu^<, \aleph^<)$-pure submodules, and their derivatives, $(\mu^<, \aleph^<)$-pure exact sequences (see Definitions 1.2 and 1.3), were first introduced in [**2**]. The $(\mu^<, \aleph^<)$-pure projective modules are new and appear here for the first time in this full generality. The special case when $\mu = \aleph_0$ and $\aleph = \aleph_0$ is the usual well-known case of pure submodules, pure exact sequences, pure injectives and pure projectives. In this special case the pure projectives appear in [**7**]. A module that has a presentation with fewer than $\mu$ generators and fewer than $\aleph$ relations is said to be $(\mu^<, \aleph^<)$-presented. Here Section 2 ends with a satisfactory characterization (Theorem 2.4); a module $M$ is $(\mu^<, \aleph^<)$-pure projective if and only if it is a direct summand of a direct sum of $(\mu^<, \aleph^<)$-presented modules.

A module $D$ is $(\mu^<, \aleph^<)$-pure injective by definition if $D$ has the injective property relative to all $(\mu^<, \aleph^<)$-pure short exact sequences. There is a satisfactory theory in the finite $\mu = \aleph = \aleph_0$ case for (ordinary) pure injectives [**5**, Vol. 1, pp. 158–174] and/or [**4**, pp. 118–122]. However, in contrast to the projective case, so far the author has been unable to develop a theory of $(\mu^<, \aleph^<)$-pure injective modules. Here Section 3 gives all that can be said in the special case $\mu = \aleph_0 \cdot \aleph$ in which case an $(\aleph_0 \cdot \aleph^<, \aleph^<)$-pure injective module is simply called $\aleph^<$-pure injective. One of the main results here is Theorem 3.1. Its proof is very different from the known finite case $\aleph = \aleph_0$. The author is unable to prove it for the general $(\mu^<, \aleph^<)$-case, and it is not clear

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