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MEASURING THE CLASSIFICATION DIFFICULTY OF COUNTABLE TORSION-FREE ABELIAN GROUPS

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1. The problem.

Question. Can we hope to classify countable torsion-free abelian groups?

Already a few remarks should be made about this question.

First of all the word "classify" is somewhat plastic in its meaning. Someone might for instance take the question to mean whether there is any sense at all in which we can understand countable torsion-free abelian groups, and I am sure "classification" takes on different hues across different guilds and mathematical specialties.

I will take the word "classify" to mean "completely classify by some class of invariants." Here I have in mind something like the Ulm invariants for countable abelian p-groups or Baer's classification for the rank one case.

Secondly one might wonder about the restriction to this particular class of groups. Here I would respond by saying that we cannot hope to classify everything, and some restrictions probably are inevitable. Abelian groups represent the topic of this conference and should be easier and more hopeful than general groups; and the choice of torsion-free further removes potentially distracting details. As for confining ourselves to the countable case, cardinality $\aleph_0$, I would mention the kinds of set theoretical complexities which can arise when one considers uncountable discrete structures. Frequently one is led into independent results and, considering subtle combinatorial properties such as the behavior of the nonstationary ideal, and even classification schemes which would be virtually perfect in the countable case, such as Ulm invariants, may begin to fail when we pass to $\aleph_1$.

Even granting these restrictions, we may want to take a skeptical stance. After all, if a classification scheme was going to be found, then

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