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THE AFFINITY OF SET THEORY AND ABELIAN GROUP THEORY

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ABSTRACT. This paper reviews the uses of set theory to solve some long-standing problems in a number of different areas of abelian group theory. In some cases the solution is an independence result (from ZFC, the ordinary axioms of set theory); in other cases the result is a theorem of ZFC proved by combinatorial methods. In the interests of breadth, and to keep within the prescribed bounds of space, some depth and detail have been sacrificed and the emphasis is on key developments in the early history of each area. In general, except for Butler groups, a cut-off date of about 1990 has been observed, except for brief references to selected later developments. Also, for reasons of space, the bibliography is not complete.

1. Slenderness and reflexivity. One of the first cases where set theory beyond ZFC had an impact on abelian group theory was the appearance of measurable cardinals in the study of slender groups. As is well known, the theory of slenderness originated with Łoś and first appeared in Fuchs' 1958 book [**37**]. While the definition of slender involves homomorphisms from $\mathbf{Z}^\omega$ into the slender group L, Theorem 47.2 of [**37**] states a property that holds for homomorphisms from $\mathbf{Z}^\kappa$ into L provided that κ is a cardinal less than the first measurable cardinal, if there is one. (This is anticipated in [**20**].) Fuchs recalls that Łoś, when he outlined the proof,

> "had a clear idea that the cardinality restriction was unavoidable. The proof in the book follows closely his outline. I was surprised, because I had never seen anything like that before and was hoping that the restriction could be removed, but as we worked on the details of the proof, it became clear to me that it was impossible to get rid of it."

The notion of a measurable cardinal was defined by Ulam in a 1930 paper [**91**]. Ulam and Tarski proved that a measurable cardinal λ

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