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# CLOSURE ALONG AN ADMISSIBLE SUBSET, SEMINORMALITY AND $T$-CLOSEDNESS

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ABSTRACT. We introduce the closure of an injective ring morphism $A \to B$ along an admissible subset $X$ of $\mathrm{Spec}\,(A)$ (an admissible subset $X$ is the spectral image of a flat epimorphism $A \to E$). Then we give a theory of seminormality and $t$-closedness along admissible subsets which extends Yanagihara's work on $S$-seminormality.

**0. Introduction.** In order to give a unified treatment for $p$-seminormality of Swan and F-closedness of Asanuma, Yanagihara introduced $S$-seminormality for rings $A$ with respect to a multiplicative subset $S$ of $A$. We refer to Yanagihara's paper for more details [**26**]. We studied $t$-closedness in two papers [**16, 17**]. This last notion is closely linked with seminormality and quasi-normality. Yanagihara's work gave us the idea to introduce $S$-$t$-closedness. However, it quickly appeared that the reason the theory works is the existence of the flat epimorphism $A \to A_S$. Thus we decided to extend the theory to any flat epimorphism. Outside localizations, flat epimorphisms appear in many contexts of commutative algebra and algebraic geometry. For instance, affine subsets of a spectrum give rise to flat epimorphisms. Evidently, such an extension brings about many technical problems but provides much more flexibility to handle results. When $A \to B$ is an injective ring morphism, we identify $A$ to a subring of $B$. If someone prefers, he could consider ring extensions $A \subset B$.

In Section 1, we begin by giving results on flat epimorphisms. Some of them come from papers of Lazard [**10**] and Raynaud [**22**]. Following Raynaud, we say that a subset $X$ of the spectrum of a commutative ring $A$ is admissible if there is a flat epimorphism $e : A \to E$ such that ${}^a e(\mathrm{Spec}\,(E)) = X$. Actually, an admissible subset $X$ determines the flat epimorphism $A \to E$ within an isomorphism. We show that for $X \subset \mathrm{Spec}\,(A)$, there is a smallest admissible subset $X_a$ containing $X$; for instance, $(V(I))_a = \mathrm{Spec}\,(A_{1+I})$ if $I$ is an ideal of a ring $A$.

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