ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 3, Fall 2003

# VALUES OF LUCAS SEQUENCES MODULO PRIMES

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ABSTRACT. Let $p$ be an odd prime, and $a, b$ be two integers. It is the purpose of the paper to determine the values of $u_{(p\pm1)/2}(a,b) \pmod p$, where $\{u_n(a,b)\}$ is the Lucas sequence given by $u_0(a,b)=0$, $u_1(a,b)=1$ and $u_{n+1}(a,b)=bu_n(a,b)-au_{n-1}(a,b)$ $(n \ge 1)$. In the case $a=-c^2$, a reciprocity law is established. As applications we obtain the criteria for $p \mid u_{(p-1)/4}(a,b)$ (if $p \equiv 1 \pmod 4$) and for $k \in Q_0(p)$ and $k \in Q_1(p)$, where $Q_0(p)$ and $Q_1(p)$ are defined as in [**10**].

**1. Introduction.** Let $a$ and $b$ be two real numbers. The Lucas sequences $\{u_n(a,b)\}$ and $\{v_n(a,b)\}$ are defined as follows:

$$\begin{gathered} u_0(a,b)=0, \quad u_1(a,b)=1, \\ u_{n+1}(a,b)=bu_n(a,b)-au_{n-1}(a,b), \quad n \ge 1; \end{gathered} \tag{1.1}$$

$$\begin{gathered} v_0(a,b)=2, \quad v_1(a,b)=b, \\ v_{n+1}(a,b)=bv_n(a,b)-av_{n-1}(a,b), \quad n \ge 1. \end{gathered} \tag{1.2}$$

It is well known that

$$\begin{aligned} u_n(a,b)=\frac{1}{\sqrt{b^2-4a}}\bigg(\Big(&\frac{b+\sqrt{b^2-4a}}{2}\Big)^n \\ &-\Big(\frac{b-\sqrt{b^2-4a}}{2}\Big)^n\bigg) \quad (b^2-4a \ne 0) \end{aligned} \tag{1.3}$$

and

$$v_n(a,b)=\left(\frac{b+\sqrt{b^2-4a}}{2}\right)^n+\left(\frac{b-\sqrt{b^2-4a}}{2}\right)^n. \tag{1.4}$$

2000 AMS *Mathematics Subject Classification.* 11B39, 11B50, 11A15, 11E25.
*Key words and phrases.* Prime, Lucas sequence, reciprocity law.
Received by the editors on November 9, 2000, and in revised form on May 30, 2001.