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AN EXTREMAL NONNEGATIVE SINE POLYNOMIAL

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ABSTRACT. For any positive integer n, the sine polynomials that are nonnegative in $[0, \pi]$ and which have the maximal derivative at the origin are determined in an explicit form. Associated cosine polynomials $K_n(\theta)$ are constructed in such a way that $\{K_n(\theta)\}$ is a summability kernel. Thus, for each p, $1 \le p \le \infty$ and for any 2π-periodic function $f \in L_p[-\pi, \pi]$, the sequence of convolutions $K_n * f$ is proved to converge to f in $L_p[-\pi, \pi]$. The pointwise and almost everywhere convergences are also consequences of our construction.

1. Introduction and statement of results. There are various reasons for the interest in the problem of constructing nonnegative trigonometric polynomials. Among them are the Gibbs phenomenon [**16**, Section 9], univalent functions and polynomials [**7**], positive Jacobi polynomial sums [**1**] and orthogonal polynomials on the unit circle [**15**].

Our study is motivated by a basic fact from the theory of Fourier series and by an intuitive observation which comes from an overview of the variety of known nonnegative trigonometric polynomials. The sequence $\{k_n(\theta)\}$ of even, nonnegative continuous 2π-periodic functions is called an *even positive kernel* if $k_n(\theta)$ are normalized by $(1/2\pi)\int_{-\pi}^{\pi} k_n(\theta)\,d\theta = 1$ and they converge locally uniformly in $(0, 2\pi)$ (uniformly on every compact subset of $(0, 2\pi)$) to zero. It is a slight modification of the definition in Katznelson's book [**8**]. In what follows we denote by $k_n * f$ the convolution $(1/2\pi)\int_{-\pi}^{\pi} k_n(t) f(\theta - t)\,dt$. It is well known that, for every 2π-periodic function $f \in L_p[-\pi, \pi]$, $1 \le p \le \infty$, the sequence of convolutions $k_n * f$ converges to f in the $L_p[-\pi, \pi]$-norm provided k_n is a sequence of even positive summability kernels. The convolutions converge also pointwise and almost everywhere. We refer to the first chapter of [**8**] for the details.

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