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ON DISTORTION UNDER QUASICONFORMAL MAPPING

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ABSTRACT. In the paper we study the range of the system of functionals $(|f(z_1)|, |f(z_2)|)$ over the class of K-quasiconformal homeomorphisms of the Riemann sphere with standard three point normalization $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$, and for different real values of z_1 and z_2. Extremal functions are given in terms of the complex dilatation dependent only on z_1, z_2. As a corollary, we derive some sharp estimates for the functional $|f(z_2)| \pm |f(z_1)|$ and $|f(z_2) \pm f(z_1)|$. The main tool of the proofs is the extremal partition of a Riemann surface by doubly connected domains.

1. Introduction. Let S_0 be a Riemann surface given by the punctured Riemann sphere $\mathbf{C} \setminus \{0, 1\}$. We shall investigate the class Q_K, of all functions $w = f(z)$ univalent and K-quasiconformal in S_0, such that $f(S_0) = S_0$ with $f(0) = 0$, $f(1) = 1$. These functions are Sobolev generalized homeomorphic solutions of the Beltrami equation

$$w_{\bar{z}} = \mu_f(z) w_z, \quad z \in S_0, \tag{1}$$

with the complex distributional derivatives

$$w_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right), \quad \text{and} \quad w_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right), \quad z = x + iy,$$

where the complex dilatation (or the Beltrami coefficient) $\mu_f(z) = f_{\bar{z}}/f_z$ is a measurable function with the norm

$$\|\mu_f\|_\infty = \text{ess sup } |\mu_f(z)| \le k < 1, \quad z \in S_0, \quad k = \frac{K-1}{K+1}.$$

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