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KERNEL GROUPS AND NONTRIVIAL GALOIS MODULE STRUCTURE OF IMAGINARY QUADRATIC FIELDS

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ABSTRACT. Let K be an algebraic number field with ring of integers $\mathcal{O}_K$, $p > 2$, be a rational prime and G the cyclic group of order p. Let Λ denote the order $\mathcal{O}_K[G]$. Let $Cl(\Lambda)$ denote the locally free class group of Λ and $D(\Lambda)$ the kernel group, the subgroup of $Cl(\Lambda)$ consisting of classes that become trivial upon extension of scalars to the maximal order. If p is unramified in K, then $D(\Lambda) = T(\Lambda)$, where $T(\Lambda)$ is the Swan subgroup of $Cl(\Lambda)$. This yields upper and lower bonds for $D(\Lambda)$. Let $R(\Lambda)$ denote the subgroup of $Cl(\Lambda)$ consisting of those classes realizable as rings of integers, $\mathcal{O}_L$, where L/K is a tame Galois extension with Galois group $Gal(L/K) \cong G$. We show under the hypotheses above that $T(\Lambda)^{(p-1)/2} \subseteq R(\Lambda) \cap D(\Lambda) \subseteq T(\Lambda)$, which yields conditions for when $T(\Lambda) = R(\Lambda) \cap D(\Lambda)$ and bounds on $R(\Lambda) \cap D(\Lambda)$. We carry out the computation for $K = \mathbf{Q}(\sqrt{-d})$, $d > 0$, $d \neq 1$ or 3. In this way we exhibit primes p for which these fields have tame Galois field extensions of degree p with nontrivial Galois module structure.

1. Introduction and subgroups of $Cl(\Lambda)$. Let K be an algebraic number field and denote its ring of algebraic integers by $\mathcal{O}_K$. Let G be a finite abelian group of order n. Let Λ denote the order $\mathcal{O}_K[G]$ in the group algebra $K[G]$. The class group of stable isomorphism classes of locally free Λ-modules is denoted by $Cl(\Lambda)$. The kernel group, $D(\Lambda)$, is the subgroup of $Cl(\Lambda)$ consisting of those classes that become trivial upon extension of scalars to the maximal order. Let $\Sigma = \sum_{g\in G} g$, then for each $r \in \mathcal{O}_K$ so that r and n are relatively prime define the Swan module $\langle r, \Sigma\rangle$ by $\langle r, \Sigma\rangle = r\Lambda + \Lambda\Sigma$. Swan modules are locally free rank one Λ-ideals and hence determine classes in $Cl(\Lambda)$ [**15**]. The set of classes of Swan modules is the Swan subgroup, denoted

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