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WEAK SYMMETRY IN NATURALLY REDUCTIVE HOMOGENEOUS NILMANIFOLDS

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ABSTRACT. We prove that within the class of naturally reductive homogeneous nilmanifolds, the notions of weak symmetry, i.e., any two points can be interchanged by an isometry, and commutativity, i.e., isometry invariant differential operators commute, are equivalent.

A connected Riemannian manifold M is said to be *weakly symmetric* if for any two points $p, q \in M$ there exists an isometry of M mapping p to q and q to p. These spaces were introduced by Selberg in the framework of his development of the trace formula, see [**12**], where it is proved that in a weakly symmetric space M, the algebra of all invariant (with respect to the full isometry group $I(M)$) differential operators on M is commutative, that is, M is a *commutative* space. Selberg asks in [**12**] whether the converse holds. The answer is negative, the known counterexamples arise in certain homogeneous nilmanifolds, so-called H-type groups, see [**8, 9**]. On the other hand, it has been proved by Akhiezer and Vinberg [**1**] that, for homogeneous spaces of reductive algebraic groups, the answer is affirmative.

In such a way, a natural question takes place: under what extra conditions on the homogeneous Riemannian manifold M, the weak symmetry is necessary for the commutativity of $I(M)$-invariant differential operators?

With such a question in mind, the first observation we make is that none of the counterexamples found is naturally reductive. A Riemannian manifold M is said to be *naturally reductive*, if there exists a transitive Lie group G of isometries with isotropy subgroup K at $p \in M$, and an $\mathrm{Ad}\,(K)$-invariant vector subspace $\mathfrak{m}$ of $\mathfrak{g}$ complementary

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