ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 1, Winter 2004

COHEN-MACAULAYNESS OF TENSOR PRODUCTS

LEILA KHATAMI AND SIAMAK YASSEMI

ABSTRACT. Let (R, m) be a commutative noetherian local ring. Suppose that M and N are finitely generated modules over R such that M has finite projective dimension and such that $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$. The main result of this note gives a condition on M which is necessary and sufficient for the tensor product of M and N to be a Cohen-Macaulay module over R, provided N is itself a Cohen-Macaulay module.

1. Introduction. Throughout this note $(R, \mathfrak{m})$ is a commutative noetherian local ring with nonzero identity and the maximal ideal $\mathfrak{m}$. By M and N we always mean nonzero finitely generated R-modules. The projective dimension of an R-module M is denoted by $\mathrm{proj.dim}\, M$.

The well-known notion "grade of M", grade M, has been introduced by Rees, see [**8**], as the least integer $t \geq 0$ such that $\mathrm{Ext}\,^t_R(M, R) \neq 0$. In [**10**], we have defined the "grade of M and N", grade (M, N), as the least integer $t \geq 0$ such that $\mathrm{Ext}\,^t_R(M, N) \neq 0$.

One of the main results of this note is Theorem 1.8, and it states:

Let N be a Cohen-Macaulay R-module, and let M be an R-module with finite projective dimension. If $\mathrm{Tor}\,^R_i(M, N) = 0$ for all $i > 0$, then $M \otimes_R N$ is Cohen-Macaulay if and only if $\mathrm{grade}\,(M, N) = \mathrm{proj.dim}\, M$.

This theorem can be considered as a generalization of the following well-known statement, cf. [**4**, Theorem 2.1.5]:

(T1) Let R be a Cohen-Macaulay local ring, and let M be a finite R-module with finite projective dimension. Then M is a Cohen-Macaulay if and only if $\mathrm{grade}\, M = \mathrm{proj.dim}\, M$.

On the other hand the following statement from Yoshida can be concluded from our result:

This research was supported in part by a grant from IPM.
1991 AMS *Mathematics Subject Classification.* 13C14, 13D45, 13H10.
Key words and phrases. Cohen-Macaulay modules.
Received by the editors on March 2, 2001, and in revised form on November 6, 2001.