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TWO THETA FUNCTION IDENTITIES AND SOME EISENSTEIN SERIES IDENTITIES OF RAMANUJAN

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ABSTRACT. In this paper we prove two general theta function identities by using the complex variable theory of elliptic functions. As applications, we provide completely new proofs of some Eisenstein series identities of Ramanujan by these two theta function identities and one famous identity of Ramanujan for the Rogers-Ramanujan continued fraction. We also derive two remarkable theta function identities relating to the modular equations of degree 5.

1. Introduction. We suppose throughout that $q = e^{2\pi i\tau}$, $\operatorname{Im}\tau > 0$; this condition ensures that all the sums and products that appear here converge. The Dedekind eta-function is defined by

$$\eta(\tau) = q^{(1/24)}(q;q)_\infty = e^{(\pi i\tau)/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau}), \tag{1.1}$$

where and throughout the paper

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1. \tag{1.2}$$

The Rogers-Ramanujan continued fraction $R(q)$ is defined by

$$R(q) = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{1.3}$$

To prove several of Ramanujan's claims on $R(q)$ that were made in his first two letters to Hardy, Watson [**16**] first proved the following

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