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RECURRENCES FOR THE PARTITION FUNCTION AND ITS RELATIVES

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ABSTRACT. For each integer $n \geq 0$, (i) $p(n) :=$ the number of unrestricted partitions of n, (ii) $q(n) :=$ the number of partitions of n into distinct parts and (iii) $q_0(n) :=$ the number of partitions of n into distinct odd parts. Conventionally, $p(0) = q(0) = q_0(0) := 1$. Presented here are: two apparently new recurrences for $p(\cdot)$ and three formulas expressing the functions $q_0(\cdot)$ and $q(\cdot)$ in terms of the function $p(\cdot)$.

1. Introduction. We begin our discussion with a definition.

Definition 1.1. As usual, $\mathbf{P} := \{1, 2, 3, \dots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$, $\mathbf{Z} := \{0, \pm 1, \pm 2, \dots\}$ and $\mathbf{Q} :=$ the set of all rational numbers. Then, for each $n \in \mathbf{N}$, (i) $p(n) :=$ the number of unrestricted partitions of n, (ii) $q(n) :=$ the number of partitions of n into distinct parts and (iii) $q_0(n) :=$ the number of partitions of n into distinct odd parts. Conventionally, $p(0) = q(0) = q_0(0) := 1$. We also adopt the convention that $p(x) = q(x) = q_0(x) := 0$ whenever $x \in \mathbf{Q} - \mathbf{N}$.

Euler's pentagonal number recurrence for the partition function $p(\cdot)$, viz.,

$$p(n) = \sum_{k \in \mathbf{P}} (-1)^{k-1} \{p(n - k(3k-1)/2) + p(n - k(3k+1)/2)\},$$

for each $n \in \mathbf{P}$, where $p(0) = 1$, has been known for more than 250 years.

Doubtless, any new recurrence for $p(\cdot)$ will always be compared with Euler's recurrence. In this paper we present two new recurrences for $p(\cdot)$, Theorems 1.2 and 1.3, below stated. Our concluding remarks

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