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CONTINUED FRACTIONS AND RESTRAINED SEQUENCES OF MÖBIUS MAPS

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ABSTRACT. Modified approximants of a continued fraction are designed to increase the rate of convergence, and these led to the notion of restrained sequences of Möbius transformations. Here we give some analytic and geometric characterizations of restrained sequences and related topics. We also give an expository account of the use of geometry, including hyperbolic geometry, in discussing restrained sequences and continued fractions.

1. Introduction. Originally a continued fraction was considered to be an expression of the form

$$\mathbf{K}(a_n|b_n) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{1.1}$$

where $a_n \neq 0$ for every n. This continued fraction was said to be convergent to $\mathbf{K}$ if the sequence of truncated (finite) continued fractions converges to $\mathbf{K}$. It is well known that one can write continued fractions in terms of Möbius transformations, and this is the point of view we take here. Indeed, we define a *continued fraction* to be a *sequence* S_n *of Möbius transformations of the form*

$$S_n = s_1 \circ \cdots \circ s_n, \quad s_n(z) = \frac{a_n}{b_n + z}, \tag{1.2}$$

where $a_n \neq 0$ for every n. The classical concept of convergence is then expressed by saying that the value of the continued fraction (1.1) is the limit (when it exists) of the sequence $S_n(0)$. We denote the complex plane by $\mathbf{C}$, and the extended complex plane by $\mathbf{C}_\infty$. Throughout this paper we shall regard Möbius transformations as acting on $\mathbf{C}_\infty$, and we reserve the notation a_n, b_n, s_n and S_n for the maps in (1.2). Note that

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