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ON THE PRESERVATION OF DIRECTION CONVEXITY UNDER DIFFERENTIATION AND INTEGRATION

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ABSTRACT. For functions which are convex in one direction, we investigate to what extent this property is preserved under differentiation and integration.

1. Introduction.

A domain $M \subset \mathbf{C}$ is said to be convex in the direction $e^{i\varphi}$ if for every $a \in \mathbf{C}$ the set

$$M \cap \left\{a + te^{i\varphi} : t \in \mathbf{R}\right\}$$

is either connected or empty. We denote by $C(\varphi)$ the family of univalent analytic functions f in the unit disk $\mathbf{D}$ with the property that $f(0) = 0$ and $f(\mathbf{D})$ is convex in the direction $e^{i\varphi}$. One of the interesting features about functions that are convex in one direction is that it is not in general so that $f \in C(\varphi)$ implies that $f(rz) \in C(\varphi)$, for $r < 1$. It was conjectured by Goodman and Saff [**2**], and later proved by Ruscheweyh and Salinas [**9**], that for $0 < r \leq \sqrt{2} - 1$ we have that $f \in C(\varphi)$ implies $f(rz) \in C(\varphi)$, but for $\sqrt{2} - 1 < r < 1$ this is not necessarily the case. In solving the Goodman-Saff conjecture, Ruscheweyh and Salinas introduced the class DCP, direction convexity preserving functions [**9**].

Definition 1.1. A function g, analytic in $\mathbf{D}$, is said to be direction convexity preserving, DCP, if for every $\varphi \in \mathbf{R}$, and every $f \in C(\varphi)$ we have $g * f \in C(\varphi)$. ($*$ denotes the Hadamard product.)

The problem of finding the largest r for which $f \in C(\varphi)$ implies $f(rz) \in C(\varphi)$ can then be formulated as finding the largest r for which the geometrical series $1/(1 - rz)$ is in DCP. Since we have $z/(1 - z)^2 * f(z) = zf'(z)$ and $\log(1 - z) * f(z) = \int_0^z (f(\zeta)/\zeta) d\zeta$, it is clear that if we can find the DCP-radius of the Koebe function

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