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WHEN ARE ASSOCIATES UNIT MULTIPLES?

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ABSTRACT. Let R be a commutative ring with identity. For $a, b \in R$ define a and b to be *associates*, denoted $a \sim b$, if $a|b$ and $b|a$, to be *strong associates*, denoted $a \approx b$, if $a = ub$ for some unit u of R, and to be *very strong associates*, denoted by $a \cong b$, if $a \sim b$ and further when $a \neq 0$, $a = rb$ implies that r is a unit. Certainly $a \cong b \Rightarrow a \approx b \Rightarrow a \sim b$. In this paper we study commutative rings R, called *strongly associate rings*, with the property that for $a, b \in R$, $a \sim b$ implies $a \approx b$ and commutative rings R, called *présimplifiable* rings, with the property that for $a, b \in R$, $a \sim b$ (or $a \approx b$) implies that $a \cong b$.

Let R be a commutative ring with identity and let $a, b \in R$. Then a and b are said to be *associates*, denoted $a \sim b$, if $a|b$ and $b|a$, or equivalently, if $Ra = Rb$. Thus if $a \sim b$, there exist $r, s \in R$ with $ra = b$ and $sb = a$ and hence $a = sra$. So if a is a regular element (i.e., nonzero divisor), $sr = 1$ and hence r and s are units. Hence if a and b are regular elements of a commutative ring R with $a \sim b$, then $a = ub$ for some $u \in U(R)$, the group of units of R. For $a, b \in R$, let us write $a \approx b$ if $a = ub$ for some $u \in U(R)$. Of course, $a \approx b$ implies $a \sim b$ for elements a and b of any commutative ring R and for an integral domain the converse is true. In [**14**], Kaplansky raised the question of when a commutative ring R satisfies the property that for all $a, b \in R$, $a \sim b$ implies $a \approx b$. He remarked that Artinian rings, principal ideal rings, and rings with $Z(R) \subseteq J(R)$ satisfy this property. (Here $Z(R)$ and $J(R)$ denote the set of zero divisors and Jacobson radical of a ring R, respectively.) But he gave two examples of commutative rings that fail to satisfy this property. Let us recall these two examples and give a third example. (1) Let $R = C([0,3])$, the ring of continuous functions on $[0,3]$. Define $a(t), b(t) \in R$ by $a(t) = b(t) = 1 - t$ on $[0,1]$, $a(t) = b(t) = 0$ on $[1,2]$, and $a(t) = -b(t) = t - 2$ on $[2,3]$. Then $a(t) \sim b(t)$ (for $c(t)a(t) = b(t)$ and $c(t)b(t) = a(t)$ where $c(t) = 1$ on $[0,1]$, $c(t) = 3 - 2t$ on $[1,2]$, and $c(t) = -1$ on $[2,3]$), but $a(t) \not\approx b(t)$.

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