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GEOMETRIC CONSTRUCTIONS ON CYCLES

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ABSTRACT. A point, plane or sphere in $\mathbf{R}^n$ can be described as a point on the Lie quadric $\Omega \subset \mathbf{P}^{n+2}$, and a geometric construction on points, planes and spheres as a map which associates a point $y \in \Omega$ to a given k-tuple $(x_1, \dots, x_k) \in \Omega^k$. In this paper the Apollonius construction is described as a map $\mathcal{A} : \mathcal{D} \to \Omega$, where $\mathcal{D}$ is a subset of Ω^{n+1}. A number of geometric constructions is obtained by composing the map $\mathcal{A}$ with Lie reflections and some other projective transformations in $\mathbf{P}^{n+2}$.

1. Introduction. A geometric construction in the space $\mathbf{R}^n$ can be viewed as a map on a set, containing geometric objects described in an appropriate way. In this paper constructions on points, and oriented hyperspheres and hyperplanes in $\mathbf{R}^n$ are considered. A suitable way to describe such geometric constructions comes from Lie geometry. In Lie geometry, oriented planes and spheres of dimension $n-1$ in $\mathbf{R}^n$, which are together called *geometric cycles*, are described as points on a quadric surface Ω in the projective space $\mathbf{P}^{n+2}$, while the angle of intersection is expressed in terms of the Lie product in $\mathbf{R}^{n+3}$. In this setting, a geometric construction on cycles can be thought of as a map from Ω^k to Ω, which associates to a given k-tuple of points, representing geometric objects in $\mathbf{R}^n$, a point in Ω, representing a solution of the construction. Lie geometry has been used to study geometric problems on circles for example in [**3, 5, 6**] and [**7**]. A thorough treatment of Lie geometry can be found in [**1**] or [**2**].

A basic example of a construction on cycles is the oriented Apollonius construction in $\mathbf{R}^n$, which asks for a sphere or plane, tangent to $(n+1)$ given spheres and planes. In [**7**], a solution of an Apollonius construction is described as a point in the intersections of a projective line in $\mathbf{P}^{n+2}$ with Ω, and a classification of Apollonius constructions,

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