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APPROXIMATION IN NON-ASPLUND SPACES

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ABSTRACT. Results on the uniform approximation of continuous functions by C^k-smooth functions on the boundary of certain convex subsets in Banach spaces which are non-Asplund are given.

1. Introduction. The uniform approximation of continuous functions by C^k-smooth maps on Banach spaces which admit C^k-smooth bump functions (C^k-smooth real-valued functions with bounded, non-empty support) has received much attention over the years, see, e.g., [**2**]. Corresponding results in non-Asplund spaces has been less common, although related work can be traced back to the seminal papers [**4**] and [**1**], while more recently, the behavior of smooth functions on non-Asplund spaces and their 'harmonic' behavior has been considered in [**2**, Theorem III.1.3 and Proposition III.1.7], and results in a similar vein are in [**3**].

A simple yet important observation is that if one is able to uniformly approximate arbitrary continuous functions on an open set G in a Banach space X via maps C^1-smooth on G, then by approximating a suitable continuous bump function on G with a C^1-smooth map on G subsequently composed with an appropriate smooth bump function on $\mathbf{R}$, one can construct a C^1-smooth bump function on X. This in turn implies that X is Asplund. Hence, for non-Asplund spaces X, it is not possible to uniformly approximate arbitrary continuous maps on open sets by functions C^1-smooth on X. This is in stark contrast to the situation for many Banach spaces which admit C^1-smooth bump functions such as reflexive spaces or, more generally, weakly compactly generated Asplund spaces.

It follows that, for non-Asplund spaces, approximation theorems are much more constrained. Nevertheless, we obtain some interesting

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