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SYMMETRIC DIOPHANTINE EQUATIONS

AJAI CHOUDHRY

ABSTRACT. In this paper we use certain properties of rational binary forms to solve several diophantine equations of the type $f(x,y) = f(u,v)$. If on applying the nonsingular linear transformation T defined by $x = \alpha u + \beta v$, $y = \gamma u + \delta v$, the binary form $\phi(x,y)$ becomes a scalar multiple of the form $\phi(u,v)$, we call $\phi(x,y)$ an eigenform of the linear transformation T. If $f(x,y) = L(x,y)\phi(x,y)$ where $\phi(x,y)$ is an eigenform of the linear transformation T and $L(x,y)$ is not an eigenform of T, the diophantine equation $f(x,y) = f(u,v)$ reduces, on making the substitution $x = m(\alpha u + \beta v)$, $y = m(\gamma u + \delta v)$, to a linear equation in the variables u and v while m is an arbitrary parameter. The solution of this linear equation readily yields a parametric solution of the original diophantine equation. We first use eigenforms to obtain parametric solutions of several general types of diophantine equations such as $L_1(x,y)Q_1^r(x,y)Q_2^s(x,y) = L_1(u,v)Q_1^r(u,v)Q_2^s(u,v)$ and $\{\Pi_{i=1}^5 L_i(x,y,z)\}Q^r(x,y,z) = \{\Pi_{i=1}^5 L_i(u,v,w)\}Q^r(u,v,w)$ where Ls and Qs denote linear and quadratic forms and r and s are arbitrary integers, and then we obtain parametric solutions of several specific diophantine equations such as the equation $f(x,y) = f(u,v)$ where $f(x,y) = x^n + x^{n-1}y + \cdots + y^n$, n being an arbitrary odd integer and the equation $x^7 + y^7 + 625z^7 = u^7 + v^7 + 625w^7$.

1. Introduction. In this paper we use certain properties of binary forms to solve several symmetric diophantine equations of the type

$$f(x,y) = f(u,v). \tag{1.1}$$

We will use Ls, Qs and Cs to denote linear, quadratic and cubic forms, respectively. All the forms considered in this paper will be assumed to be defined over the field $\mathbf{Q}$ of rational numbers. Further, reducibility of a form means reducibility over $\mathbf{Q}$.

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