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EXPLICIT ESTIMATES FOR THE RIEMANN ZETA FUNCTION

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ABSTRACT. We apply van der Corput's method of exponential sums to obtain explicit upper bounds for the Riemann zeta function on the line $\sigma = 1/2$. For example, we prove that if $t \geq e$, then $|\zeta(1/2+it)| \leq 3t^{1/6}\log t$. These results will be used in an application on primes to short intervals [**4**].

1. Introduction. It is well known that the distribution of prime numbers is related to the study of the Riemann zeta-function. For $\sigma > 1$, the Riemann zeta-function is defined to be the following infinite sum

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

where $s = \sigma + it$ with real variables σ and t.

This definition can be extended to the whole complex plane except at $s = 1$. The following definitions for $\sigma > 0$ and $s \neq 1$ can be obtained respectively by virtue of the partial and the Euler-MacLaurin summation formulae.

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{u-[u]}{u^{s+1}}\,du,$$

and

$$\zeta(s) = \frac{1}{s} + \frac{1}{2} - s\int_1^{\infty} \frac{u-[u]-1/2}{u^{s+1}}\,du.$$

For reference, one may see [**1, 8, 13, 14**]. The following formula

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)\,dx - \int_a^b f(x)\,d\biggl(x-[x]-\frac{1}{2}\biggr)$$

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