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PLANE CURVES WITH MANY POINTS OVER FINITE FIELDS

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1. Introduction. The purpose of this paper is to construct plane curves over finite fields which meet the upper bound of [**4**, Theorem 0.1], recalled below, for the number of their rational points. We also prove an irreducibility criterion for plane curves.

The upper bound of [**4**, Theorem 0.1] is the first inequality of the following Theorem in the special case of irreducible curves.

Theorem 1. *Let C be a, possibly reducible, plane algebraic curve defined over $\mathbf{F}_p$, p prime, of degree $d < p$. Suppose that C does not have a linear component defined over $\mathbf{F}_p$. Then $\#C(\mathbf{F}_p) \leq d(d+p-1)/2$. If $\#C(\mathbf{F}_p) \geq d(d+p-1)/2 - (d-1)$, then C is absolutely irreducible.*

Proof. Without loss of generality, we can assume C is reduced, for the conditions are only strengthened in this case. Let $C_1, \dots, C_m$ be the components of C over $\mathbf{F}_p$ and let d_i be the degree of C_i. By hypothesis $d_i > 1$ for all i. If C_i is absolutely irreducible, then by [**4**, Theorem 0.1], $\#C_i(\mathbf{F}_p) \leq d_i(d_i+p-1)/2$, whereas if C_i is not absolutely irreducible, then $\#C_i(\mathbf{F}_p) \leq d_i^2/4$ as follows from the proof of Lemma 3.3 of [**2**]. As $d_i^2/4 < d_i(d_i+p-1)/2$, we also get the first bound when C_i is not absolutely irreducible. Now

$$\#C(\mathbf{F}_p) \leq \sum \#C_i(\mathbf{F}_p) \leq \sum d_i(d_i+p-1)/2.$$

From $\sum d_i = d$ we get that

$$\sum d_i(d_i+p-1)/2 = d(d+p-1)/2 - \sum_{i<j} d_i d_j.$$

This, combined with the preceding inequality, gives the first statement of the theorem. To get the second statement, consider the case

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