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GENERALIZATIONS AND REFINEMENTS OF HERMITE-HADAMARD'S INEQUALITY

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ABSTRACT. In this article, with the help of the concept of the harmonic sequence of polynomials, the well known Hermite-Hadamard's inequality for convex functions is generalized to cases with bounded derivatives of nth order, including the so-called n-convex functions, from which Hermite-Hadamard's inequality is extended and refined.

1. Introduction. Let $f(x)$ be a convex function on the closed interval $[a,b]$, the well known Hermite-Hadamard's inequality [**6**] can be expressed as:

$$(1)\quad 0 \le \int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right) \le (b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(t)\,dt.$$

It is well known that Hermite-Hadamard's inequality is an important cornerstone in mathematical analysis and optimization. There is a growing literature considering its refinements and interpolations now.

A function $f(x)$ is said to be r-convex on $[a,b]$ with $r \ge 2$ if and only if $f^{(r)}(x)$ exists and $f^{(r)}(x) \ge 0$.

In terms of a trapezoidal formula and a midpoint formula for a real function $f(x)$ defined and integrable on $[a,b]$, using the first and second Euler-Maclaurin summation formulas, inequality (1) was generalized for $(2r)$-convex functions on $[a,b]$ with $r \ge 1$ in [**2**].

In this paper, for our own convenience, we adopt the following notation

$$(2)\qquad S_n=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}$$

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