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A THEOREM OF KREIN REVISITED

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ABSTRACT. M. Krein proved in [**9**] that if T is a continuous operator on a normed space leaving invariant an open cone, then its adjoint T^* has an eigenvector. We present generalizations of this result as well as some applications to C^*-algebras, operators on l_1, operators with invariant sets, contractions on Banach lattices, the Invariant Subspace Problem, and von Neumann algebras.

1. Introduction. M. Krein proved in [**9**, Theorem 3.3] that if T is a continuous operator on a normed space leaving invariant a nonempty open cone, then its adjoint T^* has an eigenvector. Krein's result has an immediate application to the Invariant Subspace problem because of the following observation. If T is a bounded operator on a Banach space and not a multiple of the identity, and $T^*f = \lambda f$, then the kernel of f is a closed nontrivial subspace of codimension 1 which is invariant under T. Moreover, $\overline{\operatorname{Range}(\lambda I - T)}$ is a closed nontrivial subspace which is proper (it is contained in the kernel of f) and hyperinvariant for T; that is, it is invariant under every operator commuting with T.

Several proofs and modifications of Krein's theorem appear in the literature, see, e.g. [**3**, Theorems 6.3 and 7.1] and [**12**, p. 315]. We prove yet another version of Krein's theorem: if T is a positive operator on an ordered normed space in which the unit ball has a dominating point, then T^* has a positive eigenvector. We deduce the original Krein's version of the theorem from this, as well as several applications and related results.

In particular, we show that if a bounded operator T on a Banach space satisfies any of the following conditions, then T^* has an eigenvector. Moreover, if the condition holds for a commutative family of operators, then the family of the adjoint operators has a common eigenvector.

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