

KARLIN'S CONJECTURE AND A QUESTION OF PÓLYA

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ABSTRACT. The paper answers an old question of Pólya involving Descartes' Rule of Signs and a related conjecture of Karlin involving the signs of Wronskians of entire functions and their derivatives. Counterexamples are given along with classes of functions for which the conjecture is valid.

0. Introduction. The purpose of this paper is to answer an old unsolved question of Pólya (c. 1934) and to resolve a related conjecture of Karlin (c. 1967). In Section 1 we state Pólya's question, Karlin's conjecture, provide some background information and recall the definitions and terminology that will be used in the sequel. For a general class of polynomials closed under differentiation, we prove in Section 2 that Descartes' Rule of Signs is equivalent to the sign regularity of certain Hankel determinants, Theorem 2.3. The counterexamples we give to Karlin's conjecture, Section 3, also provide a negative answer to Pólya's question. (While this manuscript was in preparation, Dr. Dimitar Dimitrov has kindly informed the authors that he has also obtained a counterexample to Karlin's conjecture.) In Section 4 we investigate some classes of entire functions for which Pólya's question has an affirmative answer, Theorem 4.6 and Corollary 4.8, and for which Karlin's conjecture is valid, Theorem 4.5.

1. Background information and definitions. In 1934, in connection with his investigation of the distribution of zeros of polynomials and the total positivity of certain matrices, Schoenberg [24, p. 562] cited the following question of Pólya.

Pólya's question. *Let $f(x)$ be a polynomial of degree n with only real and simple zeros, $x_1 < x_2 < \cdots < x_n$. Let $Z_{(x_n, \infty)}(f)$ denote the number of real zeros of f in the interval (x_n, ∞) , and let*

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