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EFFICIENCY FOR SELF SEMI-DIRECT PRODUCTS OF THE FREE ABELIAN MONOID ON TWO GENERATORS

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ABSTRACT. Let A and K both be copies of the free abelian monoid on two generators. For any connecting monoid homomorphism $\theta : A \to \mathrm{End}\,(K)$, let $D = K \rtimes_\theta A$ be the corresponding monoid semi-direct product. We give necessary and sufficient conditions for the efficiency of a standard presentation for D in terms of the matrix representation for θ. Let p be a prime or 0. In [**4**], necessary and sufficient conditions were given for the standard presentation of the semi-direct product of any two monoids to be p-Cockcroft. We use that result to give more explicit conditions in the special case here.

1. Introduction. Let $\mathcal{P} = [X\ ;\ \mathbf{r}]$ be a monoid presentation where a typical element $R \in \mathbf{r}$ has the form $R_+ = R_-$. Here R_+, R_- are words on X, that is, elements of the free monoid X^* on X. The *monoid defined by* $[X\ ;\ \mathbf{r}]$ is the quotient of X^* by the smallest congruence generated by $\mathbf{r}$.

We have a (Squier) graph $\Gamma = \Gamma(X;\mathbf{r})$ associated with $[X\ ;\ \mathbf{r}]$, where the vertices are the elements of X^* and the edges are the 4-tuples $e = (U, R, \varepsilon, V)$ where $U, V \in X^*$, $R \in \mathbf{r}$ and $\varepsilon = \pm 1$. The initial, terminal and inversion functions for an edge e as given above are defined by $\iota(e) = UR_\varepsilon V$, $\tau(e) = UR_{-\varepsilon}V$ and $e^{-1} = (U, R, -\varepsilon, V)$. There is a two-sided action of X^* on Γ as follows. If $W, \overline{W} \in X^*$ then, for any vertex V of Γ, $W.V.\overline{W} = WV\overline{W}$ (product in X^*) and, for any edge $e = (U, R, \varepsilon, V)$ of Γ, $W.e.\overline{W} = (WU, R, \varepsilon, V\overline{W})$. This action can be extended to the paths in Γ.

Two paths π and π' in a 2-complex are equivalent if there is a finite sequence of paths $\pi = \pi_0, \pi_1, \cdots, \pi_m = \pi'$ where for $1 \le i \le m$ the path π_i is obtained from π_{i-1} either by inserting or deleting a pair ee^{-1} of inverse edges or else by inserting or deleting a defining path for one of

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