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APPROXIMATION OF SOBOLEV-TYPE CLASSES WITH QUASI-SEMINORMS

Z. DITZIAN, V.N. KONOVALOV AND D. LEVIATAN

ABSTRACT. Since the Sobolev set W_p^r, $0 < p < 1$, in general is not contained in L_q, $0 < q \le \infty$, we limit ourselves to the set $W_p^r \cap L_\infty$, $0 < p < 1$. We prove that the Kolmogorov n-width of the latter set in L_q, $0 < q < 1$ is asymptotically 1, that is, the set cannot be approximated by n-dimensional linear manifolds in the L_q-norm. We then describe a related set, the width of which is asymptotically n^{-r}.

1. Introduction and function classes. Very little is known about the exact order of any width of nontrivial classes of functions in the L_q-metric for $0 < q < 1$. Recall that, for $1 \le p, q \le \infty$, the orders of most widths of the classical Sobolev classes W_p^r in L_q are well known. In contrast, for $0 < p < 1$, the behavior of any of the widths of these classes in L_q, $0 < q \le \infty$, are not known. In general, the class W_p^r, $0 < p < 1$, is not contained in L_q, but even if we overcome this difficulty by taking, say, the smaller set $W_p^r \cap L_\infty$, $0 < p < 1$, we will show that it cannot be approximated well in L_q for any $0 < q \le \infty$. We remind the reader that, for the approximation of $f \in L_p$, $0 < p < 1$, by polynomials and by splines with either equidistant knots or knots on the Chebyshev partition, there are known Jackson-type estimates involving the moduli of smoothness of f in the L_p-quasi-norm, see, e.g., [**1**]. However, there are no simple relations between the moduli of smoothness and the derivatives of f, if they exist. Moreover, the moduli of smoothness are not equivalent to K-functionals which are identically zero, see, e.g., [**3**, Theorem 2.1]. Thus, we introduce new classes V_p^r, $0 < p < 1$, which we feel are the proper replacement of the Sobolev classes for $0 < p < 1$, and we obtain the exact orders of their Kolmogorov, linear, and pseudo-dimensional widths in L_q, $0 < q < 1$. We also obtain for these classes exact orders of best approximation in L_q, $0 < q < 1$, by rational functions and free-knot splines.

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