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MATRICES DEFINING GORENSTEIN LATTICE IDEALS

HOSSEIN SABZROU AND FARHAD RAHMATI

ABSTRACT. We study a class of integer matrices that define Gorenstein lattice ideals. We call them Gorenstein matrices. We give a combinatorial characterization of those which are of size $(n+1) \times n$ and we relate them to the Frobenius problem in integer programming theory. We also give a necessary and sufficient condition for Gorensteinness of generic matrices which are defined in integer programming theory.

1. Introduction. Let $S = k[\mathbf{x}] := k[x_1, \dots, x_n]$ be a polynomial ring over a fixed field k. A monomial $x_1^{u_1} \cdots x_n^{u_n}$ in S is denoted by $\mathbf{x}^u$, where $u = (u_1, \dots, u_n) \in \mathrm{N}^n$. A vector $u \in \mathrm{Z}^n$ can be written uniquely as $u = u^+ - u^-$, where u^+ and u^- are positive and negative parts of u, respectively. Let $B = (b_{ij})$ be an integer $n \times d$-matrix of rank d whose columns are vectors $b_1, \dots, b_d$ in Z^n. For the lattice $\mathcal{L}_B$ in Z^n which is spanned by the columns of B, the corresponding lattice ideal in S is the binomial ideal

$$I_{\mathcal{L}_B} := \langle \mathbf{x}^{u^+} - \mathbf{x}^{u^-} \mid u \in \mathcal{L}_B \rangle.$$

The matrix B is called a defining matrix of $I_{\mathcal{L}_B}$. Such a matrix is of course not unique, but one can see easily that it is unique up to action of $SL_d(\mathrm{Z})$, that is, if B' is a second integer $n \times d$-matrix of rank d, then $I_{\mathcal{L}_B} = I_{\mathcal{L}_{B'}}$ if and only if for a unimodular matrix $T \in SL_d(\mathrm{Z})$, we have $B' = BT$.

The relationships between the matrix B and the lattice ideal $I_{\mathcal{L}_B}$ have been studied by many authors [**5, 7, 10, 15**] and [**16**]. It is well known that some numerical invariants and some algebraic properties of the lattice ideal $I_{\mathcal{L}_B}$ can be read off directly from the matrix B.

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