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ON CONVEX UNIVALENT FUNCTIONS WITH CONVEX UNIVALENT DERIVATIVES

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ABSTRACT. We study the functions

$$(*) \qquad \sum_{k=0}^{\infty} a_k \frac{(1+z)^k}{k!}, \quad a_0 \geq a_1 \geq \cdots \geq 0,$$

and show that they are either constant or convex univalent in the unit disk $\mathbf{D}$. Note that this set of functions is invariant under differentiation. Our result generalizes a previous one of Suffridge, and we verify a number of general conjectures of Suffridge and Shah & Trimble concerning functions with convex univalent derivatives for our particular cases. We also pose and discuss the conjecture that the functions $(*)$, under the further restriction that $a_1 = a_2$, actually belong to the much smaller class DCP, whose members preserve direction-convexity of univalent functions in $\mathbf{D}$, under the Hadamard product.

1. Introduction. This work was inspired by T.J. Suffridge's paper [**12**] where he studies the partial sums

$$Q_n(z) = \sum_{k=0}^{n} \frac{(1+z)^k}{k!}, \quad z \in \mathbf{C}, \quad n \in \mathbf{N}, \tag{1.1}$$

of the series $e^{1+z} = \sum_{k=0}^{\infty}(1+z)^k/k!$. His main result was that the Q_n are convex univalent in the unit disk $\mathbf{D}$. Note that $Q_n' = Q_{n-1}$ so that all derivatives of Q_n are as well convex univalent or constants. He conjectured that the normalized functions

$$C_n(z) := \frac{Q_n(z) - Q_n(0)}{Q_n'(0)} = \sum_{k=1}^{n} \left(\frac{\sum_{l=0}^{n-k} 1/l!}{\sum_{l=0}^{n-1} 1/l!} \right) \frac{z^k}{k!}, \quad n \in \mathbf{N}, \tag{1.2}$$

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