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ON RECTIFICATION OF CIRCLES AND AN EXTENSION OF BELTRAMI'S THEOREM

FARZALI IZADI

"The only perfect geometrical figures are the straight line and the circle."

Plato

ABSTRACT. The goal of this paper is to describe all local diffeomorphisms mapping a family of circles, in an open subset of $\mathbf{R}^3$, into straight lines. This paper contains two main results. The first is a complete description of the rectifiable collection of circles in $\mathbf{R}^3$ passing through one point. It turns out that to be rectifiable all circles need to pass through some other common point. The second main result is a complete description of geometries in $\mathbf{R}^3$ in which all the geodesics are circles. This is a consequence of an extension of Beltrami's theorem by replacing straight lines with circles.

Introduction. The problem that the author solved has its origin in *Nomography*[1]: how to reduce a nomogram of aligned points to a circular nomogram? In more mathematical terms: what are local diffeomorphisms that send germs of lines to germs of circles? This question was initially posed for two-dimensional nomograms by G.S. Khovanskii and solved by A.G. Khovanskii in that case, cf. [**6**]. Our result leads to a solution of the corresponding three-dimensional nomography. On the other hand, it is a continuation of Möbius' classical work that describes all transformations taking lines to lines and circles to circles. It is also related to Beltrami's investigations. By Beltrami's classical theorem, all the geometries whose geodesics are locally straight lines have constant curvature, cf. [**2, 9, 12**]. We prove that all geometries in $\mathbf{R}^3$ whose geodesics are locally circles must also have constant curvature. The similar fact in $\mathbf{R}^4$ is wrong. This was communicated to the author by Timorin, cf. [**11**] for details. We also give a complete description for all the metrics of these geometries.

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