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EXTREMAL PROBLEMS OF INTERPOLATION THEORY

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ABSTRACT. We consider problems where one seeks $m \times m$ matrix valued H^∞ functions $w(\xi)$ which satisfy interpolation constraints and a bound

$$w^*(\xi)w(\xi) \leq \rho_{\min}^2, \quad |\xi| < 1, \tag{0.1}$$

where the $m \times m$ positive semi-definite matrix $\rho_{\min}$ is minimal (no smaller than) any other matrix ρ producing such a bound. That is, if

$$w^*(\xi)w(\xi) \leq \rho, \quad |\xi| < 1, \tag{0.2}$$

and if $\rho_{\min} - \rho$ is positive semi-definite, then $\rho_{\min} = \rho$. This is an example of what we shall call a "minimal interpolation problem." Such problems are studied extensively in the book [**13**, Chapter 7]. When the bounding matrices ρ are restricted to be scalar multiples of the identity, then the problem where we extremize over them is just the classical matrix valued interpolation problem containing those of Schur and Nevalinna-Pick (which in typical cases has highly nonunique solutions). Our minimal interpolation forces tighter conditions.

In this paper we actually study a framework more general than that of Nevanlinna-Pick and Schur, and in this general context we show under some assumptions that our minimal interpolation problem, with $\rho_{\min}$ defined formally by a minimal rank condition in Definition 3.3, has a unique solution $\rho_{\min}$ and $w_{\min}(\xi)$. It is important both from applied and theoretical view points that the solution $w_{\min}(\xi)$ turns out to be a rational matrix function, indeed for the matrix Nevanlinna-Pick and Schur problems we obtain an explicit formulas generalizing those known classically.

Also in this paper we compare minimal interpolation problems to superoptimal interpolation problem, cf. [**14**] and [**11**], and see that they have very different answers. Whether one chooses super-optimal criteria or our minimal criteria in a particular situation depends on which issues are important in that situation.

The case $m = 1$ was investigated by many people with a formulation close to the one we use being found in Akhiezer

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