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THE MARKOFF-HURWITZ EQUATIONS OVER NUMBER FIELDS

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ABSTRACT. Let R be an order in a number field K, and let $\mathcal{M}_{a,n}(R)$ be the set of R-integral solutions to the Markoff-Hurwitz equation $x_1^2+\cdots+x_n^2=ax_1\cdots x_n$, where $a\in R$, $a\neq 0$, and $n\geq 3$. This set can be expressed as the orbit of a fundamental set of solutions $\mathcal{F}_{a,n}(R)$ under the action of a group of automorphisms $\mathcal{A}_{a,n}$. Hurwitz showed that $\mathcal{F}_{a,n}(\mathbf{Z})$ is always finite. Silverman showed that $\mathcal{F}_{a,3}(R)$ is often infinite if the group of units R^* in R is infinite. In this paper, we show that if R^* is infinite and K has a real imbedding, then $\mathcal{F}_{a,n}(R)$ is either empty or infinite. We also show that if K is totally complex and $n\geq 6$, then $\mathcal{F}_{a,n}(R)$ is infinite.

Introduction. The Diophantine equation

$$x_1^2+x_2^2+\cdots+x_n^2=ax_1x_2\cdots x_n \tag{1}$$

with a a nonzero integer and $n\geq 3$ is known as a Hurwitz or Markoff-Hurwitz equation. Such equations were first studied by Hurwitz [**7**] who thought of them as generalizations of the Markoff equation

$$x^2+y^2+z^2=3xyz, \tag{2}$$

which was first studied by Markoff [**8**]. The theory surrounding the Markoff equation is rich and quite extensive, but the property we are interested in here is the following: All integer solutions (x,y,z) with $0<x\leq y\leq z$ can be generated from the *fundamental solution* $(1,1,1)$ and the branching operations

$$\begin{tikzcd}[row sep=small] & (x,z,3xz-y) \\ (x,y,z) \arrow[ur, dash] \arrow[dr, dash] & \\ & (y,z,3yz-x). \end{tikzcd}$$

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